Indian Statistical Institute Midterm Examination (supplementary) 2024-2025 Analysis of Several Variables, B.Math Second Year

Time: 3 Hours Date: 07.10.2024 Maximum Marks: 100 Instructor: Jaydeb Sarkar

(1) (15 marks) True of false (with justification): There is a function $f : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = e^{xy}$$

(2) (15 marks) Use a first-order Taylor approximation to approximate the value of

f(5.999, 4.001),

where

$$f(x,y) = x\sqrt{y}$$

(3) (15 marks) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} xy\frac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Prove that f is differentiable at (0, 0).

(4) (15 marks) Let

$$w = f\Big(\frac{y-x}{xy}, \frac{z-x}{xz}\Big).$$

Assume that f is a differentiable function. Compute the value of

$$x^2\frac{\partial w}{\partial x} + y^2\frac{\partial w}{\partial y} + z^2\frac{\partial w}{\partial z}$$

(5) (15 marks) Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^2 y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Which of the following statements are true? Justify your answers:

- (a) $f_{xy}(0,0) = f_{yx}(0,0).$
- (b) f_{xy} and f_{yx} are continuous at (0,0).

(6) (25 marks) Consider the set

$$S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z \cos z\}.$$

Prove that there is a function f(x, y) such that S in the neighborhood of (0, 0) coincides with the graph z = f(x, y). Also prove that (0, 0) is a critical point of f.